On Existence, Uniqueness and Continuation of Solution of First Order Neutral Differential Equation with Piecewise Constant Argument

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Abstract: *This paper discusses the existence, uniqueness, and continuation of solution for a first order neutral differential equation with piecewise constant deviating argument.*

Key Words: *Neutral differential equations, piecewise constant deviating argument.*

1. INTRODUCTION:

The realistic models are best described by differential equations with delay arguments. The delay is either discrete or continuous or both. The hybrid systems where the delay is piecewise continuous was reported for the first time in [2] and further developed in [3], [4]. A neutral differential equation (NDE) is a differential equation where the derivative of an unknown function at present time depends not only on the history of that unknown function but also on the history of the derivative of that function. The existence, uniqueness, and continuation of solutions of these equations are studied by Driver [5], Grimm [6], Henrquez [7] and references there in, and of general equations by Hale [9, 10]. This paper discusses the existence, uniqueness, and continuation of solution for a first order neutral differential equation with piecewise constant deviating argument.
We consider the following NDE:

$$x'(t) = f\big(t, x(t), x([t]), x'([t])\big); x(0) = x_0 \,, \qquad (1)$$

Where $f \in C(K, \mathbb{R}), K \subset \mathbb{R}^4$ and [.] is the greatest integer function.

Let D denotes the class of all functions $x: J \to \mathbb{R}$, satisfying

1. $x(t)$ is continuous, for $t \in J$.
2. $x'(t)$ exists and is continuous on the intervals $[n, n+1)$, for $n = 0, 1, 2, \dots, \check{T} - 2$ and on $[\check{T} - 1, T)$.

where $\check{T} = [T] + 1, \; for\, T \neq \widetilde{[T]}, \; and\; \check{T} = T, for\; T = [T]$.

A function $x: J \to \mathbb{R}$ is said to be solution of (1) if $x \in$ D and satisfies (1) with $x'(t) = x'_+(t)$ on $t = 1, 2, 3, \dots, \check{T} - 1$.

2.PRELIMINARIES:

In this section we state lemma, definitions, and theorems which are preliminary requirements for the main results.

Lemma 2.1. ***(Ascoli-Arzela)****[1]:*
Let F be a family of functions bounded and equicontinuous at every point of an interval I. Then, every sequence of functions fn in F contains a subsequence uniformly convergent on every compact subinterval of I.

Theorem 2.2. ***(Contraction Mapping Theorem)****[8]:*
Let F be a continuous mapping of a complete metric space X into itself such that F^k is a contraction mapping of X for some positive integer k. Then F has a unique fixed point.

3. EXISTENCE AND UNIQUENESS OF SOLUTION:

In this section, we prove the existence and uniqueness of solution of (1).

Theorem 3.1. *Suppose that the following conditions are satisfied:*
(A1) $f(t, x, y, z)$ *be piecewise continuous function on*

$$S = \{0 \le t \le T, |x - x_0| \le b, |y - x_0| \le b, |z - z_0| \le c; b, c, T > 0\} \subset \mathbb{R}^4.$$

(A2) f *is bounded on* S *i.e.* $sup_{(t,x,y,z)\in S}|f| \leq M$.

(A3) $f(t,x,y,z)$ *satisfies,*

$$||f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2)|| \leq L_1||x_1 - x_2|| + L_2\, ||y_1 - y_2|| + L_3||z_1 - z_2||,$$

For $t \in [0,T]$ *and* $L_1,\ L_2,\ L_3 \geq 0$.

Then there exists $\beta \in [0,T]$ *such that for* $0 \leq t \leq \beta$, *(1) has a unique solution with initial function* $x(0) = x_0$ *at t=0.*

Proof: Let $C'(J,\mathbb{R})$ be the space of all continuous and differentiable functions.
Define $||x(t)|| = sup_{t\in[0,T]}|x(t)|$. With the above norm $C'(J,\mathbb{R})$ is a Banach space. Choose $\beta, \rho,\ \gamma \geq 1$ such that $0 \leq \beta \leq T,\ 0 \leq \rho \leq b, 0 \leq \gamma \leq c$ and $\beta M<\alpha$ where $\alpha = \min\left\{\beta, \frac{\delta}{M}\right\},\ \delta = \min\{\rho,\gamma\}$.
Let $S_1 = \{x \in C'(J,\mathbb{R}), ||x - x_0|| \leq \alpha\}$. S_1 is a closed, convex, bounded subset of the Banach space $C'(J,\mathbb{R})$. Let x_0 be any element of S_1. Define a sequence recursively on $[n, n+l]$, where $0 < l < 1$ and $n = 0, 1, 2, \ldots, \breve{\beta} - 2$ by

$$x_n^k(t) = P\left(x_n^{k-1}(t)\right) = x_n^k(n) + \int_n^t f\left(s, x_n^{k-1}(n), y_n^{k-1}(n), z_n^{k-1}(n)\right)ds.$$

$x_n^k(n) = x_n^0(n), z_n^k(n) = z_n^0(n), k = 1,2,3,\ldots,m$. We claim that $x_n^{k-1}(t)$ converges to a fixed point on $[n, n+l]$. Since $f(t,x,y,z)$ is continuous on $[n, n+l]$, the function $x_n^0(t), x_n^1(t), \ldots, x_n^m(t)$ are defined and continuous on $[n, n+l]$. Obviously $\left(t, x_n^0(t)\right) \in S_1$. Therefore, we have

$$||x_n^1(t) - x_n^0(t)|| \leq M(t-n) \leq M\beta < \alpha,$$

And hence $\left(t, x_n^1(t)\right) \in S_1$. We can show by induction that $\left|\left|x_n^k(t) - x_n^0(t)\right|\right| \leq \alpha$, which implies for $k = 2, 3, \ldots, m$, $\left(t, x_n^k(t)\right) \in S_1$. Also, it is not hard to check using Leibnitz newton theorem that, $P(x_n^{k-1}(t))$ is continuously differentiable on $[n, n+l]$.
To show the convergence of $\{x_n^m(t)\}$ we take $p_n^m(t) = x_n^m(t) - x_n^{m-1}(t)$. Consider,

$$||x_n^m(t) - x_n^{m-1}(t)|| = \left|\left|P\left(x_n^{m-1}(t)\right) - P\left(x_n^{m-2}(t)\right)\right|\right|$$
$$= \left|\left| \int_n^t f(s, x_n^{m-1}(n), y_n^{m-1}(n), z_n^{m-1}(n))ds - \int_n^t f(s, x_n^{m-2}(n), y_n^{m-2}(n), z_n^{m-2}(n))ds \right|\right|$$
$$\leq 2L(1+L)(t-n)||x_n^{m-1}(t) - x_n^{m-2}(t)||$$
$$\leq \frac{\{2L(1+L)\}^{m-1}\alpha(t-n)^m}{m!}$$

where $L = \max\{L_1, L_2, L_3\}$.

$$||p_n^m(t)|| \leq \frac{\{2L(1+L)\}^{m-1}\alpha}{m!} \qquad (2)$$

Next we consider an infinite series of the form

$$x(t) = x_0 + \sum_{i=1}^{\infty} p_n^i(t). \qquad (3)$$

The m[th] partial sum of this series is $x_n^m(t)$, i.e.

$$x_n^m(t) = x_0 + \sum_{i=1}^{m} p_n^i(t). \qquad (4)$$

The sequence $\{x_n^m(t)\}$ converges iff (4) converges. From inequality (2), we have

$$x_0 + \sum_{i=1}^{m} ||p_n^i(t)|| \leq x_0 + \sum_{i=1}^{m} \frac{\{2L(1+L)\}^{i-1}\alpha}{i!}. \qquad (5)$$

As $i \to \infty, \frac{\{2L(1+L)\}^{i-1}\alpha}{i!} \to 0$. Therefore, $\lim_{m\to\infty} x_n^m(t) = x_n(t)$. From the uniform convergence of $x_n^m(t)$ to $x_n(t)$ and the continuity of the function $f(t,x,y,z)$ on S_1, it follows that $f(t, x_n^m, y_n^m, z_n^m) \to f(t, x_n, y_n, z_n)$ uniformly on $[n, n+l]$ *as* $m \to \infty$. Hence, $x_n(t)$ is a solution of (1) on $[n, n+l]$.
Now, for $n = 0$ we get solution $x_0(t)$. We extend the interval of existence from [0,1] to [1,2] as $l \to 1$. We apply the local existence theorem on this interval with initial condition $x_0(1) = x_1$. This way the interval of existence can be extended to $[0, \beta]$.
In order to prove the uniqueness, let $u(t)$ be any other solution of (1) and $x_n(t) \neq u_n(t)$ on $t \in [n, n+l]$.

$$||x_n(t) - u_n(t)|| \leq ||x_{n-1}(t) - u_{n-1}(t)|| + \left|\left| \int_n^t f(s, x_n(s), x_n(n), x'_n(n))ds - \int_n^t f\left(s, u_n(s), u_n(n), u'_n(n)\right)ds \right|\right|$$
$$\leq \alpha + \{2L(1+L)\} \int_n^t ||x_n(s) - u_n(s)||ds.$$

Applying Gronwall's inequality, we get

$$||x_n(t) - u_n(t)|| \leq \alpha e^{2L(1+L)t}.$$

Hence choosing α suitably, it is easy to see that $||x(t) - u(t)|| < \epsilon \; for \; t \in [n, n+l]$. Hence $x(t) = u(t)$.

Remark 3.2. *Under the condition of Theorem 3.1, the solution x(t) can be extended to the boundary of the set S.*

In the next theorem, we are going to relax the Lipschitz condition (A3) on f and obtain the result for existence of solution.

Theorem 3.3. *Let the function f(t,x,y,z) be continuous and bounded in the strip*

$$S_2 = \{0 \leq t \leq \beta, |x| < \infty, \beta > 0\}.$$

Then, the initial value problem (1) has at least one solution x(t) defined on the interval $[0, \beta]$.

Proof: We define a sequence of functions $\{x_n^k(t)\} on \; [n, n+l]$, where $n = 0,1,2,3, \dots \check{\beta} - 2 \; and \; on \; [\check{\beta} - 1, \beta]$ as follows:
For, $t = n, x_n^k(t) = x_{n-1}(n)$
and $t \in [n, n+l]$, $x_n^k(t) = x_{n-1}(n) + \int_n^k f(\, s, x_n^{k-1}(s), y_n^{k-1}(n), z_n^{k-1}(n))ds$ (6)

From the definition for $n = 0$ we have $x_n^k = x_0$ at $t = 0$. This definition we use to define x_n^k on the interval $t \in [0,1]$ as $l \to 1$, which can be further extended to the next interval $t \in [1,2]$. By continuing the process $x_n^k(t)$ is well defined for the interval $[n, n+l]$. Also, it is not hard to check using Leibnitz newton theorem that, $x_n^k(t))$ is continuously differentiable on $[n, n+l]$.

Also, we have f bounded on S_2. Therefore, for $(t, x) \in S$ we have $|f(t, x, y, z)| \leq M$; where $M > 0$ is some constant. Using (6) for $t_1, t_2 \in [n, n+l]$, we get

$$\left|x_n^k(t_1) - x_n^k(t_2)\right| \leq M|t_1 - t_2|,$$

Which shows the sequence $\{x_n^k(t)\}$ is equicontinuous on $[n, n+l]$.
Consequently,

$$\left|x_{nk}^{(t)}\right| \leq |x_{n-1}(n)| + M\beta,$$

Which shows that $\{x_n^k(t)\}$ is uniformly bounded on $[n, n+l]$. Hence by Ascoli-Arzela's lemma there exists a subsequence $\left\{x_n^{k_p}\right\}, p = 0,1,2, \dots$which converges uniformly to continuous function $x(t)$ on $[n, n+l]$. So we have

$$x_n^{k_p}(t) = x_{n-1}(n) + \int_n^t f\left(s, x_n^{k_p}(n), y_n^{k_p}(n), z_n^{k_p}(n)\right) ds.$$

As $p \to \infty$, we have

$$x_n(t) = x_{n-1}(n) + \int_n^t f(s, x_n, y_n, z_n)ds, \qquad t \in [n, n+l].$$

This shows that the initial value problem (1) has a solution on $[n, n+l]$. Now, for $n = 0$ we get solution $x_0(t)$. We extend the interval of existence from [0,1] to [1,2] as $l \to 1$. We apply the local existence theorem on this interval with initial condition $x_0(1) = x_{1.}$ This way the interval of existence can be extended to $[0, \beta]$.

Corollary 3.4. *If f is monotonically nondecreasing in y and z for each fixed t on S, then IVP (1) has a unique solution on* $[0, \beta]$.

Proof: Let, there exist two solutions *u(t)* and *v(t)* on $[0, \beta]$. We show that *u(t) = v(t.).* Let us suppose not. Then there exists $t \in [n, n+l]$ such that $u_n(t_n) = v_n(t_n)$ and $u_n(t) < v_n(t), for \; t \in [n, t_n)$.
Then, $u_n(n) < v_n(n)$ and , $u'_n(n) < v'_n(n)$.
For sufficiently small $h < 0$. It follows $u'(t) \geq v'(t), t \in [n, t_n)$,
which gives,

$$f\big(t, u_n(t_n), u_n(n), u'_n(n)\big) \geq f\big(t, v(t_n), v_n(n), v'_n(n)\big),$$

For $t \in [n, t_n)$, which is acontradiction to the fact that f is nondecreasing on $[n, n+l]$.
Hence $u(t) = v(t), t \in [n, n+l]$. Consequently, $u(t) = v(t) for \; t \in [0, \beta]$.

4 CONTINUOUS DEPENDENCE OF SOLUTION

When we are describing the physical process using IVP for differential equations we desire that if there is some error in the initial data then it should not effect the solution very much i.e. we seek the continuous dependence of solution of IVP on initial condition.

In this section, we along with (1) are considering the following system:

$$y'(t) = f\big(t, y(t), y([t]), y'([t])\big) + g\big(t, y(t), y([t]), y'([t])\big), \qquad (7)$$

Where $g \in C(S, \mathbb{R})$.

Theorem 4.1. *Consider the system (7) where $f(t, y(t), y([t]), y'([t]))$ satisfies the assumptions of Theorem 3.1 and $g(t, y(t), y([t]), y'([t]))$ is an integral function of t. Then, (7) has a unique solution y(t) on $[0, \beta^*]$with initial condition $y(0) = y_0$ for t = 0. Let x(t) be the unique solution of (1) on $[0, \beta]$ with initial condition $x(0) = x_0$ for t=0. Let $\beta_1 = min\{\beta, \beta^*\}$, then for $\epsilon > 0$, there exists a $\delta(\epsilon, g) > 0$ such that $|x_0 - y_0| < \delta$ and $|g(t, y(t), y([t]), y'([t])| < \delta$ implies $|x(t) - y(t)| < \epsilon, t \in [0, \beta_1]$.*

Proof: For $t \in [n, n + l], where\ n = 0, 1, 2, \dots, \breve{\beta_1} - 2\ and\ [\breve{\beta_1} - 1, \breve{\beta_1}]$. Let $x_n(t)$ and $y_n(t)$ be solutions on $[n, n + l]$. Then

$$\begin{aligned} |x_n(t) - y_n(t)| \le\ & |x_{n-1}(n) - y_{n-1}(n)| \\ & + \int_n^t |f\big(s, x_n(s), x_n(n), x'_n(n)\big) - f(s, y_n(s), y_n(n), y'_n(n)|ds \\ & + \int_n^t |g(s, y_n(s), y_n(n), y'_n(n)|ds, \ \ t \in [n, n + l]. \\ \le\ & \delta + 2L(1 + L)\int_n^t |x_n(s) - y_n(s)|ds + \int_n^t |g(s, y_n(s), y_n(n), y'_n(n)|ds|, \\ \le\ & \delta(1 + \beta_1) + 2L(1 + L)\int_n^t |x_n(s) - y_n(s)|ds. \end{aligned}$$

Applying Gronwall's inequality, we get

$$|x_n(t) - y_n(t)| \le \delta(1 + \beta_1)e^{2L(1+L)t}.$$

Hence choosing δ suitably, it is easy to see that $|x(t) - y(t)| < \epsilon\ for\ t \in [0, \beta_1]$. This completes the proof of the theorem.

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