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# Mathematical Analysis of Solution of Differential Equation Longitudinal Dispersion in Unsaturated Media

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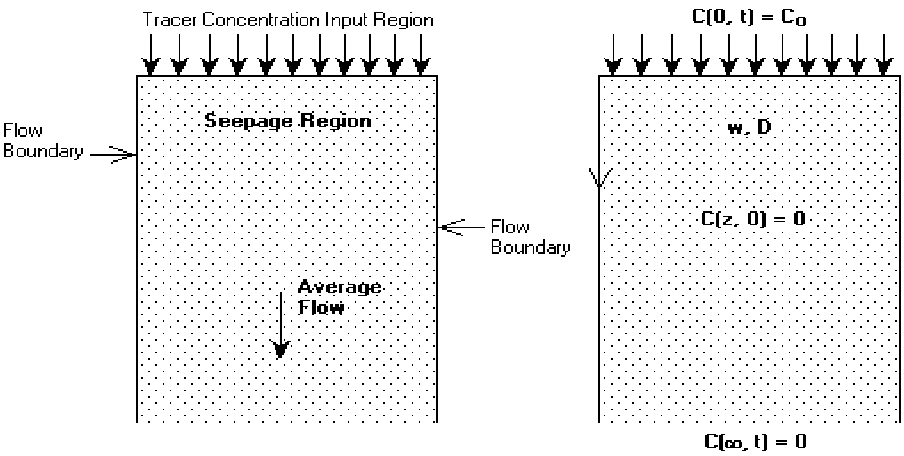
## 1. INTRODUCTION:

Most of the investigators use the coordinate transformation (*x* - *ut*) in order to solve the equation for dispersion of a moving fluid in porous media. Further, the boundary conditions $C = 0$ at $x$ = ¥ and $C = C_0$ at $x$ = – ¥ for $t > 0$ are used, which results in a symmetrical concentration distribution. This paper presents a solution of differential equation of longitudinal dispersion with variable coefficients in a finite domain. It is then shown that this solution approaches that given by symmetrical boundary conditions, provided the dispersion coefficient *D* is small and the region near the source is not considered. The solution is obtained for the diffusion model of longitudinal mixing with variable coefficients in a finite length initially solute free domain.

In the beginning, homogeneous domain is studied for dependent advection-diffusion along with uniform flow. Solution also obtained for the uniform velocity has been considered spatially dependent due to the heterogeneity of the domain and the dispersion is considered proportional to the square of the velocity. The velocity is linearly interpolated and small increment in it along the finite domain. The input condition is considered continuous of uniform and of increasing nature both. The solutions are obtained for both the domains by using Duhamel's theorem and integral solution technique. The effects of the dependency of dispersion with time and the heterogeneity of the domain on the solute transport are studied separately with the help of graphs.

## 2. Physical Layout of the Model:




Because mass is conserved, the governing differential equation is determined to be

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left( D(x,t)\frac{\partial C}{\partial x} - u(x,t)\,C \right)$$

where $C$ is solute concentration at position $x$ along the longitudinal direction at time $t$, $D$ is dispersion coefficient and $u$ is the average velocity of fluid or superficial velocity. To study the temporally dependent solute dispersion of a uniform input concentration of continuous nature in an initially solute free finite domain, we consider

$$D(x,t) = D_0 f(mt) \;\; and \;\; u(x,t) = u_0$$

When *m* is a coefficient whose dimension is inverse of the time variable. Thus *f(mt)* is an expression in non-dimensional variable *(mt)*. The expression of $f(mt) = 1$ for $m = 0$ or $t = 0$. The former case represents the uniform solute dispersion and the latter case represents the initial dispersion. The coefficients $D_0$ and $u_0$ in equation (2) may be defined as initial dispersion coefficient and uniform flow velocity, respectively. Thus the partial differential equation (1) along with initial condition and boundary conditions may be written as:

$$\frac{\partial C}{\partial t} = D_0 f(mt) \frac{\partial^2 C}{\partial x^2} - u_0 \frac{\partial C}{\partial x}$$

Initially, saturated flow of fluid of concentration, $C = 0$, takes place in the medium. At $t = 0$, the concentration of the plane source is instantaneously changed to $C = C_0$. Thus, the appropriate boundary conditions are

$$\left.\begin{aligned} C(x,0) &= 0 \quad & x \geq 0 \\ C(0,t) &= C_0 \quad & t \geq 0 \\ C(\infty,t) &= 0 \quad & t \geq 0 \end{aligned}\right\}$$

The problem then is to characterize the concentration as a function of *x* and *t*. where the input condition is assumed at the origin and a second type or flux type homogeneous condition is assumed. $C_0$ is initial concentration. To reduce equation (3) to a more familiar form, we take the moving coordinates as

$$C(x,t) = \Gamma(x,t)\exp\left(\frac{u_0 x}{2D_0 f(mt)} - \left\{\frac{u_0^2 t}{4D_0 f(mt)}\right\}\right)$$

Substituting the above equation gives
$$\frac{\partial \Gamma}{\partial t} = D_0 f(mt) \frac{\partial^2 \Gamma}{\partial x^2}$$

The initial and boundary conditions transforms to

$$\left.\begin{aligned} \Gamma(0,t) &= C_0 \exp\left[\frac{u_0^2 t}{4D_0 f(mt)}\right] \quad & t \geq 0 \\ \Gamma(x,0) &= 0 \quad & x \geq 0 \\ \Gamma(\infty,t) &= 0 \quad & t \geq 0 \end{aligned}\right\}$$

The problem then is to characterize the concentration as a function of z and t, where the input condition is assumed at the origin and a second type or flux type homogeneous condition is assumed. To reduce the governing equation into a Fick's law (diffusion equation) , we have considered the moving coordinates

$$C(z,\ t) = \Gamma(z,\ t)\,Exp\left[\frac{wz}{2D} - \frac{w^2 t}{4D}\right]$$

The Advection-dispersion equations is of the familiar form
$$\frac{\partial \Gamma}{\partial t} = D \frac{\partial^2 \Gamma}{\partial z^2}$$

The initial and boundary condition transform to

$$\left.\begin{aligned} \Gamma(0,\ t) &= C_0\ Exp\left[\frac{w^2 t}{4D}\right] \quad & t \geq 0 \\ \Gamma(z,\ 0) &= 0 \quad & z \geq 0 \\ \Gamma(\infty,\ t) &= 0 \quad & t \geq 0 \end{aligned}\right\}$$

If C=F(x, y, z, t) s the solution of the diffusion equation for semi-infinite media in which the initial concentration is zero and its surface is maintained at concentration unity, then the solution of the problem in which the surface is maintained at temperature f(t) is (Duhamel's Theorem)

$$C = \int_0^t \phi(\lambda) \frac{\partial}{\partial t} F(x,y,z,t-\lambda)\,d\lambda$$

This theorem is used principally for heat conduction problems, but the above has been specialized to fit this specific case of interest. Consider now the problem in which initial concentration is zero and the boundary is maintained at concentration unity. The boundary conditions are

$$\left.\begin{aligned} \Gamma(x,0) &= 0 \quad & x \geq 0 \\ \Gamma(0,t) &= 1 \quad & t \geq 0 \\ \Gamma(\infty,t) &= 0 \quad & t \geq 0 \end{aligned}\right\}$$

By using Duhamel's theorem and integral transform, we obtain the final solution as

$$\frac{C(x,t)}{C_0\left(1-e^{-\gamma t}\right)}=\frac{1}{2}\left\{erfc\left(\frac{x-ut}{2\sqrt{D_0\,f(mt)\,t}}\right)+\exp\left(\frac{ux}{D_0\,f(mt)}\right)erfc\left(\frac{x+ut}{2\sqrt{D_0\,f(mt)\,t}}\right)\right\}$$

Re-substitute the value of the $u$ in terms of $u_0$, we get

$$\frac{C(x,t)}{C_0\left(1-e^{-\gamma t}\right)}=\frac{1}{2}\left\{erfc\left(\frac{x-u_0t}{2\sqrt{D_0\,f(mt)\,t}}\right)+\exp\left(\frac{u_0x}{D_0\,f(mt)}\right)erfc\left(\frac{x+u_0t}{2\sqrt{D_0\,f(mt)\,t}}\right)\right\} \quad (1)$$

where boundaries are symmetrical the solution of the problem is given by the first term the above equation. The second term is this equation is thus due to the asymmetric boundary imposed in the more general problem. However, it should be noted also that if a point a great distance away from the source is considered, then it is possible to approximate the boundary condition by

$$C(-\infty,t)=C_0$$

which leads a symmetrical solution Spatially dependent dispersion along non-uniform flow

The advection-diffusion equation assumes the form $\dfrac{\partial C}{\partial t}=D_0(1+ax)^2\dfrac{\partial^2 C}{\partial x^2}-u_0(1+ax)\dfrac{\partial C}{\partial x}$

Then the desired solution may be written as

$$\frac{C(x,t)}{C_0}=\frac{1}{2}\left\{erfc\left(\frac{x-u_0(1+ax)t}{2(1+ax)\sqrt{D_0t}}\right)+\exp\left(\frac{u_0(1+ax)x}{D_0(1+ax)^2}\right)erfc\left(\frac{x+u_0(1+ax)t}{2(1+ax)\sqrt{D_0t}}\right)\right\} \quad (2)$$

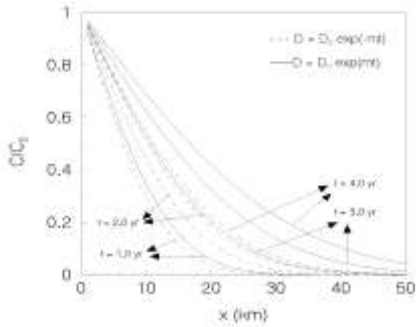


Figure 1: Temporal dependent solute dispersion uniform flow of uniform input described by solution.

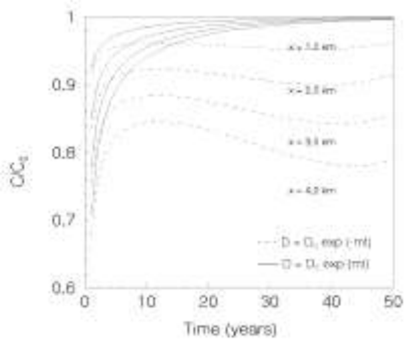


Figure 2: Break through curve for dispersion along along with uniform flow.

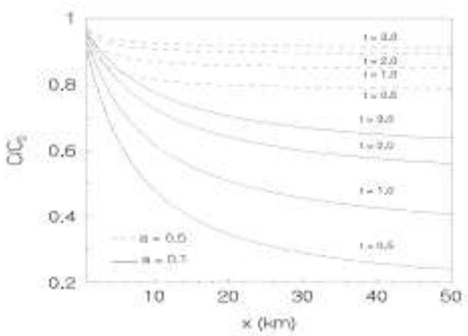


Figure 3: Spatially dependent solute dispersion along non-uniform flow of uniform input described by solution (equation 2).

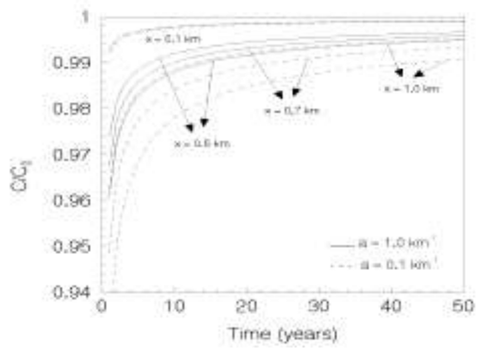


Figure 4: Break through curve for dispersion along with non-uniform flow.

## 3. RESULT AND DISCUSSIONS:

Concentration values are evaluated from the four analytical solutions discussed in a finite domain at times $t$ (years) = 1.0, 2.0, 3.0 and 4.0, for input values $C_0$ = 1.0, $u_0$ = 0.11 (km/year), $D_0$ = 50 (km$^2$/year). Figures 1 represents temporal dependent concentration dispersion pattern of uniform input and input of increasing nature, respectively along a uniform flow through a homogeneous medium, described by the analytical solutions, equation (1), respectively. In figure 1, the uniform input concentration value is 1.0 at all times and the concentration value at $x = 0$ increases with time. Thus the respective input boundary conditions are satisfied. In the figure the dotted curves represents the solutions for an expression $f(mt) = \exp(-mt)$ which is of decreasing nature. In the figures the solid curve represents the respective solutions at $t = 1.0$ (year), for another expression $f(mt) = \exp(mt)$, which is of increasing nature. It may be observed that in case of uniform input the concentration value at a particular position is higher for the latter expression of $f(mt)$ than that for the former expression of $f(mt)$. The difference increases with the distance along the domain. But in case of an input concentration of increasing nature its value is less for increasing nature of $f(mt)$ than that for decreasing nature of $f(mt)$. This trend is of diminishing nature up to $x = 2.0$, beyond which the trend reverses. For all the curves drawn in figure 1, a value $m(\text{year})-1 = 1.0$ is chosen. Both the analytical solutions of section 2 may be solved using other expressions of $f(mt)$ which satisfy the conditions stated at the outset of the section 2.

- The distribution is symmetrical for values of $x$ chosen some distance from the source. An example of break through curves obtained for dispersion in a cylindrical vertical column is shown as Figure 2. The theoretical curve was obtained by neglecting the second term of equation (1).

- Figure 3 gives the concentration values evaluated from analytical solutions (eq. 2) for spatially dependent dispersion of uniform input and input of increasing nature, respectively, along non-uniform flow, through an heterogeneous domain. The solid curves in figure 3 represent the solution in which a value $a = 1.0$ (km$^{-1}$) is taken. Using expressions it may be evaluated that due to the heterogeneity of the medium, the velocity $u$ varies from a value of 0.11 (km/year) to a value of 0.22 (km/year) and dispersion $D$ varies from a value of 0.21 (km/year) to a value of 0.42 (km/year), along the domain $0 \leq x(\text{km}) \leq 1$. This figure also shows the effect of heterogeneity on the dispersion pattern. A dotted curve is drawn for the value $a = 0.1$ (km$^{-1}$). It may be observed that the concentration values evaluated from the solution (eq. 2) along a medium of lesser heterogeneity (which introduces lesser variation in velocity and dispersion along the column), are slightly higher than those at the respective positions of a medium of higher heterogeneity, near the origin but decrease at faster rate as the other end of the medium is approached. This comparison is done at $t = 2.0$ (year). This value is chosen to ensure that the factor $(u_0 - aD_0)$ in condition remains positive for the values chosen for $u_0$ and $D_0$. The distribution is symmetrical for values of $x$ chosen some distance from the source. A break through curve is obtained for dispersion in for different depth as shown in Figure 4. The theoretical curve was obtained by neglecting the second term of equation (eq. 2).

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