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# Dominator Coloring of Some Cycle Related Graphs

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***Abstract:*** A graph has a dominator coloring if it has a proper coloring in which each vertex of the graph dominates every vertex of some color class. The dominator chromatic number is the minimum number of color classes in a dominator coloring of a graph. We present here graphs for which $\chi_d(G) = \chi(G) + \gamma(G)$ *and* $\chi_d(G) = \chi(G) + \gamma(G) - 1$.

**Keywords:** *coloring, domination number, dominator coloring, switching of cycle, splitting of cycle.*

## 1. INTRODUCTION:

We begin with simple, finite, connected and undirected graph with vertex set V(G) and edge set E(G). A proper coloring $f : V(G) \to \mathrm{N}$ assigns colors $1, 2, 3, \ldots$ to the vertices of $G$, such that adjacent vertices are assigned different colors. Then $f(v)$ is called the color of $f(v)$. The chromatic number $\chi(G)$ is the smallest integer k such that G admits a proper coloring using k colors. The set of vertices with a particular color is called color class.

There are many variants of colorings like a-coloring, b-coloring, total coloring, dominator coloring etc. The present paper is aimed to report some results on dominator coloring which was introduced in recent past by Gera *et al.* (1). The dominator coloring is a frontier between the concepts, coloring of graphs and domination in graphs.

A set $S \subseteq V(G)$ is called dominating set if every vertex $v \in V$ is either an element of S or is adjacent to an element of S. A dominating set S is a minimal dominating set if no proper subset $S'$ of S is a dominating set. The domination number $\gamma(G)$ is the minimum cardinality of minimal dominating set of G. A dominator coloring of a graph G is a proper coloring in which each vertex of the graph dominates every vertex of some color class. The dominator chromatic number $\chi_d(G)$ is the minimum number of color classes in a dominator coloring of a graph G. This concept was introduced by Gera (2) and the same author has invented dominator coloring for bipartite graphs in (3). In

(2) the author has posed the problem for what graph does $\chi_d(G) = \chi(G) + \gamma(G)$. We have found an affirmative answer to this question and it has been shown that for the splitting graph of $C_n$, $\chi_d(G) = \chi(G) + \gamma(G)$ when n is even while $\chi_d(G) = \chi(G) + \gamma(G) - 1$ when n is odd. The dominator coloring for bipartite graphs has been investigated by Gera (3). The concept of dominator coloring is also explored by Kavitha and David (4,5,6).

**PROPOSITION 1.1:(7)**

If G is a disconnected graph with components $G_1, G_2, \ldots, G_k$ with $k \geq 2$, then $\max_{i \in \{1,2,\ldots,k\}} \chi_d(G_i) + k - 1 \leq \chi_d(G) \leq \sum_{i=1}^{k} \chi_d(G_i)$

**PROPOSITION 1.2:(2)**

The star graph $K_{1,n}$ has $\chi_d(K_{1,n}) = 2$.

**PROPOSITION 1.3:(7)**

Let G be a connected unicyclic graph. Then $\chi_d(G) = \chi(G)$ if and only if G is isomorphic to $C_3$ or $C_4$ or $C_5$ or the graph obtained from $C_3$ by attaching any number of leaves at one or two vertices of $C_3$.

**PROPOSITION 1.4:(7)**

Let G be a connected graph. Then $\max\{\chi(G), \gamma(G)\} \leq \chi_d(G) \leq \chi(G) + \gamma(G)$.

**PROPOSITION 1.5:(7)**

For any bipartite graph G, $\gamma(G) \leq \chi_d(G) \leq \gamma(G) + 2$.

**DEFINITION 1.6: (8)**

The switching of a vertex v of G means removing all the edges incident to v and adding edges joining to every vertex which is not adjacent to v in G. We denote the resultant graph by $\tilde{G}$.

If $\tilde{C}_n$ is the graph obtained by switching of an arbitrary vertex v of $C_n$ then we have the following results.

**PROPOSITION 1.7:(9)**

$$\chi\left(\tilde{C}_n\right)=\begin{cases}2, & n=4\\ 3, & n\geq 5\end{cases}$$

**PROPOSITION 1.8:(10)**

$$\gamma\left(\tilde{C}_n\right)=\begin{cases}1, & n=4\\ 2, & n=5,6,7\\ 3, & n\geq 8\end{cases}$$

**DEFINITION 1.9: (11)**

The splitting graph $S'(G)$ of a graph G is obtained by adding a new vertex $v'$ corresponding to each vertex v of G such that $N(v)=N(v')$, where N(v) and $N(v')$ are the neighbourhood sets of v and $v'$ respectively in $S'(G)$.

**PROPOSITION 1.10:(9)**

$$\chi\left(S'(C_n)\right)=\begin{cases}2, & when\ n\ is\ even\\ 3, & when\ n\ is\ odd\end{cases}$$

**PROPOSITION 1.11:(11)**

For all $n\geq 3$

$$\gamma\left(S'(C_n)\right)=\begin{cases}\frac{n}{2} & if \quad n\equiv 0\ (\bmod 4)\\ \frac{n+1}{2} & if \quad n\equiv 1,3\ (\bmod 4)\\ \frac{n+2}{2} & if \quad n\equiv 2\ (\bmod 4)\end{cases}$$

## 2. MAIN RESULTS:

**THEOREM 2.1:** $$\chi_d(\tilde{C}_n)=\begin{cases}3, & n=3\\ 2, & n=4\\ 3, & n=5\\ 4, & n=6\\ 5, & n\geq 7\end{cases}$$

**Proof:** Let $v_1,v_2,\ldots,v_n$ be the vertices of $\tilde{C}_n$. Without loss of generality we switch the vertex $v_1$ and denote the resultant graph by $\tilde{C}_n$. We consider the following cases.

**Case-1: For n = 3**

The graph $\tilde{C}_3$ is the disconnected graph. Then according to Proposition-1.1, $3\leq \chi_d\left(\tilde{C}_3\right)\leq 3 \Rightarrow \chi_d\left(\tilde{C}_3\right)=3$.

**Case-2: For n = 4**

In this case the graph $\tilde{C}_4$ is same as $K_{1,3}$ .Then by Proposition-1.2, $\chi_d\left(K_{1,3}\right)=2=\chi_d\left(\tilde{C}_4\right)$.

**Case-3: For n = 5**

The graph $\tilde{C}_5$ is a unicyclic graph and it can be interpreted as two leaves attached to $C_3$. Then according to Proposition-1.3, $\chi\left(\tilde{C}_5\right)= \chi_d\left(\tilde{C}_5\right)$. Since by Proposition-1.7, $\chi\left(\tilde{C}_5\right)=3$. Thus $\chi(\tilde{C}_5)= \chi_d(\tilde{C}_5)=3$.

**Case-4: For n = 6**

By Proposition-1.7, $\chi\left(\tilde{C}_6\right)=3$ and by Proposition-1.8, $\gamma\left(\tilde{C}_6\right)=2$. Then according to Proposition-1.4, $3 \le \chi_d\left(\tilde{C}_6\right) \le 5$.

If $\chi_d\left(\tilde{C}_6\right)=3$. Then to assign dominator coloring to the vertices of $\tilde{C}_6$ we consider the color class $c=\{1, 2, 3\}$ and define the color function $f:V \to \{1, 2, 3\}$ as $f(v_1)=f(v_2)=f(v_6)=1,\ f(v_3)=2,\ f(v_4)=3,\ f(v_5)=2$. This coloring does not satisfy the condition of dominator coloring as the vertex $v_1$ dominates color class 2 and 3, $v_3$ dominates color class 3, $v_4$ dominates color class 2, $v_5$ dominates color class 3, but the vertices $v_2$ and $v_6$ do not dominate any of the color classes as they are the pendant vertices. Therefore, they will dominate their own color classes or to the color classes of adjacent vertices. But the vertex $v_2$ will not dominate its own color class 1 as it is not the only vertex with color 1 and also it will not dominate the color class 2 of adjacent vertex $v_3$ as all the vertices assigned by color 2 are not adjacent to the vertex $v_2$, similarly with vertex $v_6$. Thus, one more color is required to color the graph $\tilde{C}_6$.

Suppose $\chi_d\left(\tilde{C}_6\right)=4$, then for dominator coloring consider the color class $c=\{1, 2, 3, 4\}$ and define the color function $f:V \to \{1, 2, 3, 4\}$ as $f(v_1)=f(v_2)=f(v_6)=1,\ f(v_3)=2,\ f(v_4)=3,\ f(v_5)=4$. Thus f satisfies the condition to be a dominator coloring of $\tilde{C}_6$ as the vertex $v_1$ dominates color classes 2, 3 and 4 the vertex $v_2$ dominates color class 2, the vertex $v_3$ dominates color class 3, $v_4$ dominates color class 4, vertex $v_5$ dominates color class 3 and 4 while the vertex $v_3$ dominates color class 4. Hence $\chi_d\left(\tilde{C}_6\right)=4$.

**Case-5: For n = 7**

By Proposition-1.7, $\chi\left(\tilde{C}_7\right)=3$ and by Proposition-1.8, $\gamma\left(\tilde{C}_7\right)=2$. Then according to Proposition-1.4, $3 \le \chi_d\left(\tilde{C}_7\right) \le 5$.

If $\chi_d\left(\tilde{C}_7\right)=3$. Consider the color class $c=\{1, 2, 3\}$ and define the color function $f:V \to \{1, 2, 3\}$ as $f(v_1)=f(v_2)=f(v_7)=1,\ f(v_3)=2,\ f(v_4)=3,\ f(v_5)=2,\ f(v_6)=3$. This coloring does not satisfy the condition of dominator coloring as the vertex $v_1$ dominates color class 2 and 3, $v_4$ dominates color class 2, $v_5$ dominates color class 3, while the vertices $v_2$, $v_3$ $v_6$, $v_7$ do not dominate any of the color classes as the vertex $v_2$ and $v_7$ are the pendant vertices. Therefore, they will dominate their own color classes or to the color classes of adjacent vertices. But the vertices $v_2$ and $v_7$ will not dominate their own color class 1 as they are not the only vertices with color 1 and also they will not dominate the color classes 2 and 3 of adjacent vertices $v_3$ and $v_6$ as all the vertices assigned by colors 2 and 3 are not adjacent to the vertices $v_2$ and $v_7$. And according to the nature of the graph $\tilde{C}_7$ the vertices $v_3$ and $v_6$ does not dominate any one of the color classes. Thus $\chi_d(\tilde{C}_7) \neq 3$.

If $\chi_d\left(\tilde{C}_7\right)=4$ then for dominator coloring consider the color class $c=\{1, 2, 3, 4\}$ and define the color function $f:V \to \{1, 2, 3, 4\}$ as $f(v_1)=f(v_2)=f(v_7)=1,\ f(v_3)=2,\ f(v_4)=3,\ f(v_5)=4,\ f(v_6)=2$. This coloring does not satisfy the condition of dominator coloring as the vertex $v_1$ dominates color classes 2, 3 and 4, vertex $v_3$ dominates color class 3, vertex $v_4$ dominates color class 4, vertex $v_5$ dominates color class 3, vertex $v_6$ dominates color class 4 while the vertices $v_2$ and $v_7$ does not dominate any of the color classes as the vertex $v_2$ and $v_7$ are the pendant vertices. Therefore, they will dominate their own color classes or to the color classes of adjacent vertices. But the vertices $v_2$ and $v_7$ will not dominate its own color class 1 as it is not the only vertex with color 1 and also it will not dominate the color class 2 of adjacent vertices as they are not adjacent to all the vertices assigned by color 2. Thus $\chi_d(\tilde{C}_7) \neq 4$.

Suppose $\chi_d(\tilde{C}_7)=5$. Consider the color class $c=\{1, 2, 3, 4, 5\}$ and define the color function $f:V \to \{1, 2, 3, 4, 5\}$ as $f(v_1)=1,\ f(v_2)=2,\ f(v_7)=3,\ f(v_3)=f(v_5)=4,\ f(v_4)=f(v_6)=5$. Thus f satisfies the condition to be a dominator coloring of $\tilde{C}_7$ as the vertex $v_1$ dominates color classes 4 and 5, $v_2$ dominates its own color class 2, vertices $v_3, v_4, v_5$ and $v_6$ dominate color class 1 and the vertex $v_7$ dominates its own color class 3. Hence $\chi_d(\tilde{C}_7)=5$.

**Case-6:** For $n > 7$

We color the vertices $v_1, v_2, \ldots v_n$ for the graph $\tilde{C}_n$ as follows. $f(v_1) = 1,\ f(v_2) = 3,\ f(v_n) = 2,\ f(v_{2k+3}) = 4,\ f(v_{2k+4}) = 5,$ where $k = 0,1,2,3,\ldots$.

Thus f satisfies the condition to be a dominator coloring of $\tilde{C}_n$ as the vertex $v_1$ dominate the color classes 4 and 5, $v_2$ dominates its own color class 3, $v_n$ dominates its own color class 2 and the vertices $v_{2k+3}$ and $v_{2k+4}$ $(k = 0, 1, 2, \ldots)$ dominates color class 1. Hence $\chi_d(\tilde{C}_n) = 5,\ n \ldots 7$.

**THEOREM 2.2:** $\chi_d\left[S'(C_4)\right] = 4$

**Proof:** Let $\{v_1, v_2, v_3, v_4\}$ be the vertices and $\{e_1, e_2, e_3, e_4\}$ be the edges of cycle $C_n$. Moreover $\{u_1, u_2, u_3, u_4\}$ be the newly added vertices corresponding to the vertices $\{v_1, v_2, v_3, v_4\}$ to construct the graph $S'(C_4)$.

For the graph $S'(C_4)$, according to Proposition-1.10, $\chi\left[S'(C_4)\right] = 2$ and by Proposition-1.11, $\gamma\left[S'(C_4)\right] = 2$. Then according to Proposition-1.4, $2 ,, \chi_d\left[S'(C_4)\right] ,, 4$. Since the graph $S'(C_4)$ is the bipartite graph. Hence $\chi_d\left[S'(C_4)\right] = 4$.

**THEOREM 2.3:**

*For odd* $n \ldots 3$

$$\chi_d[S'(C_n)] = \chi[S'(C_n)] + \gamma[S'(C_n)] - 1$$

$$\chi_d\left[S'(C_n)\right] = \chi\left[S'(C_n)\right] + \gamma\left[S'(C_n)\right] - 1,\ for\ n = 3, 5, 6, 7$$

**Proof:** Let $\{v_1, v_2, \ldots v_n\}$ be the vertices and $\{e_1, e_2, \ldots e_n\}$ be the edges of cycle $C_n$. Moreover $\{u_1, u_2, \ldots u_n\}$ be the newly added vertices corresponding to the vertices $\{v_1, v_2, \ldots v_n\}$ to construct the graph $S'(C_n)$.

We initiate the coloring by assigning $\gamma\left[S'(C_n)\right]$ number of colors to the vertices of graph $S'(C_n)$ which belongs to D. Next, we assign the colors to the remaining vertices using $\chi\left[S'(C_n)\right] - 1$ number of colors.

This proper coloring pattern give rise to a dominator coloring for the respective graphs. Hence $\chi_d[S'(C_n)] = \chi[S'(C_n)] + \gamma[S'(C_n)] - 1$.

**EXAMPLE 2.4:**

To illustrate Theorem-2.3 we consider the graph $S'(C_9)$ as shown in Figure 1.

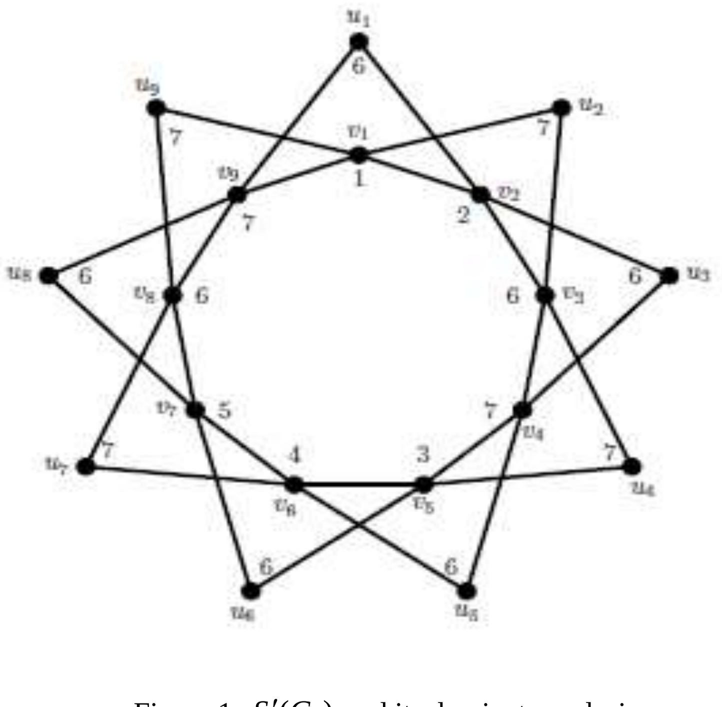


Figure 1: $S'(C_9)$ and its dominator coloring

For the graph $S'(C_9)$, $\gamma[S'(C_9)]=5$ and $D=\{v_1, v_2, v_5, v_6, v_7\}$. First, we assign the 5 distinct colors to the vertices of D respectively. Since $\chi[S'(C_9)]=3$. Next, we assign proper coloring to the remaining vertices using $\chi[S'(C_9)]-1=2$ colors. This proper coloring pattern give rise to a dominator coloring for the graph $S'(C_9)$.

Hence $\chi_d[S'(C_9)]=\chi[S'(C_9)]+\gamma[S'(C_9)]-1=3+5-1=7$.

**THEOREM 2.5:**
For even $n \ldots 6$

$\chi_d[S'(C_n)]=\chi[S'(C_n)]+\gamma[S'(C_n)]$

**Proof:** We continue with the terminology and notations used in Theorem-2.3.
We begin the coloring by assigning $\gamma\left[S'(C_n)\right]$ number of colors to each vertex v of graph $S'(C_n)$ which belongs to D. Next, we assign the colors to the vertices of N(v) using $\chi\left[S'(C_n)\right]$ number of colors.
This proper coloring procedure give rise to a dominator coloring for the respective graphs. Hence $\chi_d[S'(C_n)]=\chi[S'(C_n)]+\gamma[S'(C_n)]$.

**EXAMPLE 2.6:**
To illustrate Theorem-2.5 we consider the graph $S'(C_{10})$ as shown in Figure 2.

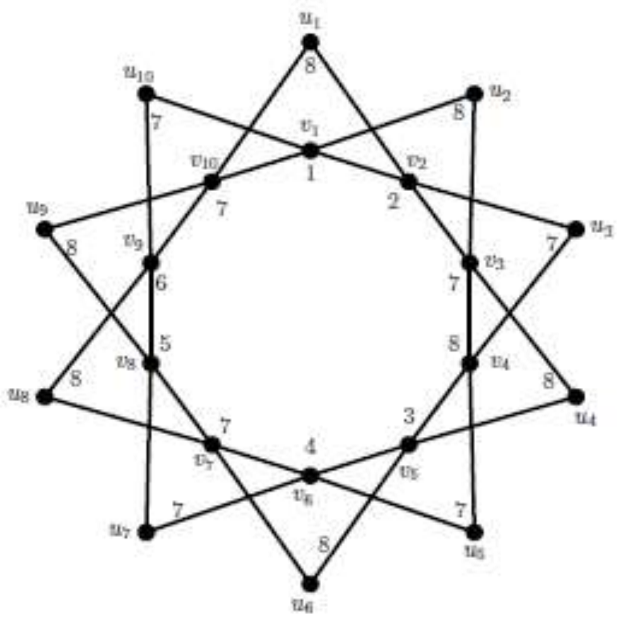


Figure 2: $S'(C_{10})$ and its dominator coloring

For the graph $S'(C_{10})$, $D=\{v_1, v_2, v_5, v_6, v_8, v_9\}$ is a dominating set with minimum cardinality. Hence $\gamma[S'(C_{10})]=6$. First, we assign the $6$ distinct colors to the vertices of $\gamma$ - set respectively. Next, we assign proper coloring to the remaining 14 vertices using $\chi[S'(C_{10})]=2$ colors. This proper coloring pattern give rise to a dominator coloring for the graph $S'(C_{10})$. Hence $\chi_d\left[S'(C_{10})\right]=\gamma\left[S'(C_{10})\right]+2$.

**3. CONCLUSION:** The dominator coloring is a variant of proper coloring of graphs. Here we have investigated the dominator chromatic number for the larger graphs obtained from $C_n$ by means of graph operations like switching of a vertex in $C_n$ as well as splitting graph of $C_n$.

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